

Nichols Algebras and Quantum Principal Bundles

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Abstract

A general procedure for constructing Yetter–Drinfeld modules from quantum principal bundles is introduced. As an application a Yetter–Drinfeld structure is put on the cotangent space of the Heckenberger–Kolb calculi of the quantum Grassmannians. For the special case of quantum projective space the associated braiding is shown to be non-diagonal and of Hecke type. Moreover, its Nichols algebra is shown to be finite-dimensional and equal to the anti-holomorphic part of the total differential calculus.

1 Introduction

For any braided vector space (V, c) , which is to say a vector space V together with a linear map $c : V \otimes V \rightarrow V \otimes V$ satisfying the braid relation, there is a naturally associated graded algebra called a Nichols algebra. In the case where c is the anti-flip map $c(v \otimes w) = -w \otimes v$, this reduces to the exterior algebra of V . Hence Nichols algebras can be considered as a far-reaching generalisation of exterior algebras. Since they were first considered by Nichols [15], Nichols algebras have reappeared in a number of diverse areas. They were rediscovered by Woronowicz [21] and Majid [9] in the theory of differential calculi, and play a central role in Lusztig’s braided Hopf algebra approach to the study of quantised enveloping algebras. Later they arose naturally in the work of Fomin and Kirrilov [3], Majid [11], and Bazlov [2] on the Schubert calculus of flag manifolds. Moreover, they are fundamental to the classification program of Andruskiewitsch and Schneider [1, 7] for pointed Hopf algebras, an area that has seen major advances in recent years.

The appearance of Nichols algebras in the work of Woronowicz came during his attempt to construct higher forms from first-order bicovariant calculi using their Yetter–Drinfeld structure. Such a construction cannot be applied to calculi which are not bicovariant, such as, for example, the distinguished Heckenberger–Kolb calculi for the irreducible quantum flag manifolds. Recent work [6] by Krähmer and Tucker–Simmons showed however that the anti-holomorphic parts of these calculi could be described as the Koszul complex of a braided symmetric algebra as introduced by Berenstein and Zwicknagl.

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This allowed for the construction of q -deformations of the Dolbeault–Dirac operator of the corresponding classical spaces, and indicates that the calculi may admit a Nichols algebra description.

In this paper we show that for the special case of quantum projective space, the antiholomorphic calculus does indeed admit a Nichols algebra description. This is done by introducing a general procedure for constructing Yetter–Drinfeld modules from quantum principal bundles, allowing us to construct a Yetter–Drinfeld structure on the cotangent space of the Heckenberger–Kolb calculi of the quantum Grassmannians. For the special case of quantum projective space, the Nichols algebra is shown to be equal to the antiholomorphic calculus, hence showing that it has the structure of a finite dimensional braided Hopf algebra in a category of Yetter–Drinfeld modules. Moreover, the braiding is shown to be non-diagonal and of Hecke type, hence implying that the algebra is Koszul.

In subsequent work it is hoped to more thoroughly examine the Nichols algebra of the quantum Grassmannians, to determine in particular if it is finite dimensional, diagonal, or Hecke. An expected byproduct is an explicit description of the cotangent space relations of Heckenberger–Kolb calculus, an essential piece of information for investigating the noncommutative Kähler geometry of the quantum Grassmannians.

The paper is organised as follows. In Section 2 we recall basic preliminaries on Yetter–Drinfeld modules and quantum principle bundles. In Section 3 we show that under a natural assumption on the base space calculus, one can always put a Yetter–Drinfeld structure on the cotangent space. In Section 4 the motivating examples of the quantum Grassmannians are presented.

2 Preliminaries

In this section we recall some basic algebraic material about Nichols algebras and Yetter–Drinfeld modules. We then present the basic theory of differential calculi and quantum principal bundles.

2.1 Nicholas Algebras, Yetter–Drinfeld Modules, and Coquasitriangular Structures

2.1.1 Nicholas Algebras

Let $\pi : \mathbb{B}_n \rightarrow \mathbb{S}_n$ be the canonical surjective group homomorphism of the braid group of n elements onto the symmetric group of n elements. As is well known, there exists a set theoretic section of this projection $S : \mathbb{B}_n \rightarrow \mathbb{S}_n$ called the *Matsumoto section*, which is not a group homomorphism but which does satisfy $s(t_i t_{i+1}) = s(t_i) s(t_{i+1})$, where t_i are the standard generators of \mathbb{B}_n .

Let (V, σ) be a braided vector space, which is to say a vector space V together with a linear map $\sigma : V \otimes V \rightarrow V \otimes V$ satisfying the braid equation. This implies an associated

representation ρ_n of the braid group \mathbb{B}_\times on $V^{\otimes n}$. The *Nichols algebra* of (V, σ) is the algebra

$$\mathfrak{B}(V) := \bigoplus_{n \in \mathbb{N}_0} \mathfrak{B}^n(V), \quad \text{where } \mathfrak{B}^n(V) := \mathcal{T}^n(V) / \ker \left(\sum_{\pi \in S_n} \rho(s(\pi)) \right).$$

A braided vector space (V, σ) is called *diagonal* if V admits a basis $\{e_i\}$ for which $\sigma(e_i \otimes e_j) = \lambda_{ij} e_j \otimes e_i$, for some set of scalars λ_{ij} . A braided vector space is said to be of *Hecke type* if, for some $\lambda \in \mathbb{C}$,

$$(\sigma - \lambda \text{id})(\sigma + 1) = 0.$$

Proposition 2.1 [1] *For $\mathcal{B}(\mathcal{V})$ the Nichols algebra of a braiding of Hecke type, with λ not a root of unity, it holds that*

1. $\mathcal{B}(\mathcal{V})$ is a Koszul algebra, which is to say a graded algebra admitting a minimal graded free resolution,
2. the relations of $\mathcal{B}(\mathcal{V})$ are generated in degree 2, which is to say, generated by $\ker(A_2)$.

2.1.2 Yetter–Drinfeld Modules

A very important class of braided vector spaces is the *Yetter–Drinfeld modules* V over a Hopf algebra H , which are those H -modules V , with an action \triangleleft , and a H -comodule structure such that

$$v_0 \triangleleft h_{(1)} \otimes v_{(1)} \triangleleft h_{(2)} = (v \triangleleft h_{(2)})_{(0)} \otimes h_{(1)}(v \triangleleft h_{(2)})_{(1)}.$$

We denote the category of Yetter–Drinfeld modules, endowed with its obvious monoidal structure, by \mathcal{YD}_H^H . The braiding for the category is defined by

$$\sigma : V \otimes W \rightarrow W \otimes V, \quad v \otimes w \mapsto w_0 \otimes v \triangleleft w_{(1)}.$$

A *braided bialgebra* in a braided monoidal category (\mathcal{C}, σ) is an algebra object H in the category which is also a coalgebra object such that the coproduct and counit morphisms are algebra maps with respect to the braided tensor product

$$(m \otimes m) \circ (\text{id} \otimes \sigma \otimes \text{id}) : (H \otimes H) \times (H \otimes H) \rightarrow H \otimes H.$$

As is well known, a Nichols algebra $\mathcal{B}(V)$ in \mathcal{YD}_H^H is a braided Hopf algebra with respect to the morphisms

$$\Delta(v) = 1 \otimes v + v \otimes 1, \quad \varepsilon(v) = 0, \quad v \in V.$$

2.1.3 Coquasitriangular Structures

We now consider another important type of braided vector space coming from a special type of bilinear map on a Hopf algebra called a coquasitriangular structure. It can be considered as dual to the notion of a quasitriangular structure.

Definition 2.2. We say that a Hopf algebra G is *coquasi-triangular* if it is equipped with a convolution-invertible linear map $r : G \otimes G \rightarrow \mathbb{C}$ obeying, for all $f, g, h \in G$, the relations

$$r(fg \otimes h) = r(f \otimes h_{(1)})r(g \otimes h_{(2)}), \quad r(f \otimes gh) = r(f_{(1)} \otimes h)r(f_{(2)} \otimes g),$$

$$g_{(1)}f_{(1)}r(f_{(2)} \otimes g_{(2)}) = r(f_{(1)} \otimes g_{(1)})f_{(2)}g_{(2)}, \quad r(f \otimes 1) = r(1 \otimes f) = \varepsilon(f).$$

For two comodules $V, W \in \text{Mod}^H$ a braiding in the category is given by

$$\sigma : V \otimes W \rightarrow W \otimes V, \quad v \otimes w \mapsto w_{(0)} \otimes v_{(0)}r(v_{(1)} \otimes w_{(1)}).$$

2.2 Quantum Principal Bundles

2.2.1 Principal Comodule Algebras

For a right H -comodule V with coaction Δ_R , we say that an element $v \in V$ is *coinvariant* if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by $V^{\text{co}H}$ and call it the *coinvariant subspace* of the coaction. (We define a coinvariant subspace of a left-coaction analogously.) For a right H -comodule algebra P , its coinvariant subspace $M := P^{\text{co}H}$ is clearly a subalgebra of P . If the mapping

$$\text{can} = (m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) : P \otimes_M P \rightarrow P \otimes H,$$

is an isomorphism, then we say that P is a *Hopf-Galois extension* of H .

Recall that P is said to be *faithfully flat* as a right module over M if the functor $P \otimes_M - : {}_M\text{Mod} \rightarrow {}_{\mathbb{C}}\text{Mod}$, from the category of left M -modules to the category of complex vector spaces, maps a sequence to an exact sequence if and only if the original sequence is exact. The definition of faithfully flat as a right M -module is analogous. A *principal right H -comodule algebra* is a right H -comodule algebra (P, Δ_R) such that P is a Hopf-Galois extension of $M := P^{\text{co}H}$ and P is faithfully flat as a right and left M -module.

For H a Hopf algebra, a *homogeneous* right H -coaction on G is a coaction of the form $(\text{id} \otimes \pi) \circ \Delta$, where $\pi : G \rightarrow H$ is a surjective Hopf algebra map. A *quantum homogeneous space* $M := G^H$ is the coinvariant subspace of such a coaction. It is easy to see [13] that every faithfully flat quantum homogeneous space G is a Hopf-Galois extension of M , and hence a principal comodule algebra. We call such a principal comodule a *homogenous principal comodule algebra*.

Proposition 2.3 *A strong bicovariant splitting map is a unital linear map $i : H \rightarrow G$ splitting the projection $\pi : G \rightarrow H$ such that*

$$(i \otimes \text{id}) \circ \Delta = \Delta_R \circ i, \quad (\text{id} \otimes i) \circ \Delta = \Delta_L \circ i. \quad (1)$$

The existence of such a map implies that $M := P^{\text{co}(H)}$ is a homogeneous principal comodule algebra.

For a more detailed discussion of bicovariant splitting maps and their relation to the theory of noncommutative principal connections see [10].

2.2.2 First-Order Differential Calculi

A *first-order differential calculus* over a unital algebra A is a pair (Ω^1, d) , where Ω^1 is an A - A -bimodule and $d : A \rightarrow \Omega^1$ is a linear map for which the *Leibniz rule* holds

$$d(ab) = a(db) + (da)b, \quad a, b \in A,$$

and for which $\Omega^1 = \text{span}_{\mathbb{C}}\{adb \mid a, b \in A\}$. We call an element of Ω^1 a *one-form*. A *morphism* between two first-order differential calculi $(\Omega^1(A), d_\Omega)$ and $(\Gamma^1(A), d_\Gamma)$ is a bimodule map $\varphi : \Omega^1(A) \rightarrow \Gamma^1(A)$ such that $\varphi \circ d_\Omega = d_\Gamma$. Note that when a morphism exists it is unique. The *direct sum* of two first-order calculi $(\Omega^1(A), d_\Omega)$ and $(\Gamma^1(A), d_\Gamma)$ is the calculus $(\Omega^1(A) \oplus \Gamma^1(A), d_\Omega + d_\Gamma)$. We say that a first-order calculus is *connected* if $\ker(d) = \mathbb{C}1$.

The *universal first-order differential calculus* over A is the pair $(\Omega_u^1(A), d_u)$, where $\Omega_u^1(A)$ is the kernel of the product map $m : A \otimes A \rightarrow A$ endowed with the obvious bimodule structure, and d_u is defined by

$$d_u : A \rightarrow \Omega_u^1(A), \quad a \mapsto 1 \otimes a - a \otimes 1.$$

By [21, Proposition 1.1] there exists a surjective morphism from $\Omega_u^1(A)$ onto any other calculus over A .

We say that a first-order differential calculus $\Omega^1(M)$ over a quantum homogeneous space M is *left covariant* if there exists a (necessarily unique) map $\Delta_L : \Omega^1(M) \rightarrow G \otimes \Omega^1(M)$ such that $\Delta_L(mdn) = \Delta(m)(\text{id} \otimes d)\Delta(n)$, for $m, n \in M$. Any covariant calculus $\Omega^1(M)$ is naturally an object in ${}^G_M\text{Mod}_M$. Using the surjection $\Omega_u^1(M) \rightarrow \Omega^1(M)$, it can be shown that there exists a subobject $I \subseteq M^+$ (where M^+ is considered as an object in ${}^H\text{Mod}_M$ in the obvious way) such that an isomorphism is given by

$$\Phi(\Omega^1(M)) \rightarrow M^+/I, \quad [dm] \mapsto [m^+].$$

This association defines a bijection between covariant first-order calculi and sub-objects of M^+ .

Finally, let us consider the case of the trivial quantum homogeneous space $\varepsilon : G \rightarrow \mathbb{C}$. Here we additionally have an obvious notion of *right covariance* for a calculus with

respect to the coproduct coaction Δ_R . If a calculus is both left and right covariant and satisfies $(\text{id} \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes \text{id}) \circ \Delta_R$, then we say that it is *bicovariant*. It was shown in [21, Theorem 1.8] that a left-covariant calculus is bicovariant if and only if the corresponding ideal I is invariant under the *(right) adjoint coaction* $\text{Ad} : G \rightarrow G \otimes G$, defined by $\text{Ad}(g) := g_{(2)} \otimes S(g_{(1)})g_{(3)}$.

2.2.3 Differential Calculi

A graded algebra $\mathcal{A} = \bigoplus_{k \in \mathbb{N}_0} \mathcal{A}_k$, together with a degree 1 map d , is called a *differential graded algebra* if d is a *graded derivation*, which is to say, if it satisfies the *graded Leibniz rule*

$$d(\alpha\beta) = d(\alpha)\beta + (-1)^k \alpha d\beta, \quad \text{for all } \alpha \in \mathcal{A}^k, \beta \in \mathcal{A}.$$

The operator d is called the *differential* of the differential algebra.

Definition 2.4. A *differential calculus* over an algebra A is a differential algebra (Ω^\bullet, d) such that $\Omega^0 = A$, and $\Omega^k = \text{span}_{\mathbb{C}}\{a_0 da_1 \wedge \cdots \wedge da_k \mid a_0, \dots, a_k \in A\}$.

We say that a differential calculus $(\Gamma^\bullet, d_\Gamma)$ *extends* a first-order calculus (Ω^1, d_Ω) if there exists an isomorphism $\varphi : (\Omega^1, d_\Omega) \rightarrow (\Gamma^1, d_\Gamma)$. It can be shown that any first-order calculus admits an extension Ω^\bullet which is *maximal* in the sense that there exists a unique morphism from Ω^\bullet onto any other extension of Ω^1 , where the definition of differential calculus morphism is the obvious extension of the first-order definition [18, §2.5]. We call this extension the *maximal prolongation* of the first-order calculus.

2.2.4 Quantum Principal Bundles

For a right H -comodule algebra (P, Δ_R) with $M := P^{\text{co}(H)}$, it can be shown that the extension $M \hookrightarrow P$ being Hopf–Galois is equivalent to exactness of the sequence

$$0 \longrightarrow P\Omega_u^1(M)P \xrightarrow{\iota} \Omega_u^1(P) \xrightarrow{\text{ver}} P \otimes H^+ \longrightarrow 0, \quad (2)$$

where $\Omega_u^1(M)$ is the restriction of $\Omega_u^1(P)$ to M , ι is the inclusion map, and ver is the restriction of can to $\Omega^1(P)$. The following definition generalises this sequence to non-universal calculi.

Definition 2.5. A *quantum principal bundle* is a Hopf–Galois extension $\Delta_R : P \rightarrow P \otimes H$ together with a sub-bimodule $N \subseteq \Omega_u^1(P)$ which is coinvariant under the right H -coaction Δ_R and for which there exists an Ad_R -coinvariant right ideal $I \subseteq H^+$ satisfying $\text{ver}(N) = G \otimes I$.

Let us denote by $\Omega^1(P)$ the first-order calculus corresponding to N_P , by $\Omega^1(M)$ the restriction of $\Omega^1(P)$ to M , and finally $\Lambda_H^1 := H^+/I$. The quantum principal bundle definition implies that a well-defined exact sequence is given by

$$0 \longrightarrow P\Omega^1(M)P \xrightarrow{\iota} \Omega^1(P) \xrightarrow{\text{ver}} P \otimes \Lambda_H^1 \longrightarrow 0. \quad (3)$$

3 Braided Vector Spaces and Quantum Principal Bundles

Let $M = G^{\text{co}H}$ be a homogeneous principal comodule algebra with strong bicovariant splitting map $i : H \rightarrow G$. Assume also that it is endowed with a quantum principal bundle structure with respect to which $\Omega^1(M)$ is an object in ${}^G_M\text{Mod}_0$.

3.1 Yetter–Drinfeld Modules

The following theorem shows how to associated to any quantum principal bundle the structure of a Yetter–Drinfeld module, and hence a braiding and a Nichols algebra.

Theorem 3.1 *A right H -action is defined by*

$$\triangleleft : V_M \otimes H \rightarrow V_M, \quad ([m], h) \mapsto [mi(h)],$$

and this module structure is independent of the choice of i , and a Yetter–Drinfeld module is given by the triple $(V_M, \triangleleft, \text{Ad}_R)$.

Proof. For the action to be well-defined it should hold that

$$\triangleleft(i(hh') - i(h)i(h')) = 0. \quad (4)$$

Now $\pi(i(hh') - i(h)i(h')) = 0$, and it follows from faithful flatness that the kernel of $\pi = M^+G$. Hence, since M^+ acts trivially on V_M we must have that (4) holds and that i defines an action of H on V_M .

Note that Ad_R acts on $m \in M$ as $\text{Ad}_R(m) = m_{(2)} \otimes \pi(S(m_{(1)})m_{(3)}) = m_{(2)} \otimes S(m_{(1)})$. Hence, since M is a right subcomodule of G , the map Ad_R does indeed restrict to a H -coaction on V_M . We now show that Ad_R and \triangleleft define a Yetter–Drinfeld structure on V_M . Note first that

$$\begin{aligned} ([m] \triangleleft h_{(2)})_{(0)} \otimes h_{(1)}([m] \triangleleft h_{(2)})_{(1)} &= [mi(h_{(2)})]_{(0)} \otimes h_{(1)}[mi(h_{(2)})]_{(1)} \\ &= [mi(h_{(2)})_{(2)}]_{(0)} \otimes h_{(1)}\pi\left(S(m_{(1)}i(h_{(2)})_{(1)})m_{(3)}i(h_{(2)})_{(3)}\right) \\ &= [m_{(2)}i(h_{(2)})_{(2)}] \otimes h_{(1)}(S \circ \pi \circ i(h_{(2)})_{(1)})(\pi \circ S(m_{(1)}))(\pi \circ i(h_{(2)})_{(3)}). \end{aligned}$$

Note now that (2.3) implies the identity

$$\pi \circ i(h)_{(1)} \otimes i(h)_{(2)} \otimes \pi \circ i(h)_{(3)} = h_{(1)} \otimes i(h_{(2)}) \otimes h_{(3)}.$$

Combining this with the previous calculation we see that, as required,

$$\begin{aligned} ([m] \triangleleft h_{(2)})_{(0)} \otimes h_{(1)}([m] \triangleleft h_{(2)})_{(1)} &= [m_{(2)}i(h_{(3)})] \otimes h_{(1)}S(h_{(2)})(\pi \circ S(m_{(1)}))h_{(4)} \\ &= [m_{(2)}] \triangleleft h_{(1)} \otimes (\pi \circ S(m_{(1)}))h_{(2)} \\ &= [m]_{(0)} \triangleleft h_{(1)} \otimes [m]_{(1)} \triangleleft h_{(2)}. \end{aligned}$$

Finally, we show that the Yetter–Drinfeld structure is independent of the choice of bico-variant splitting map. Let i' be a second splitting map and note that since

$$\pi(i(h) - i'(h)) = \pi \circ i(h) - \pi \circ i'(h) = h - h = 0,$$

we must have $\text{Im}(i - i') \subseteq \ker(\pi) = M^+G$. Hence since we have assumed $\Omega^1(M)M^+ \subseteq M^+\Omega^1(M)$, it must hold that

$$0 = [m(i(h) - i'(h))] = [mi(h)] - [mi'(h)],$$

implying that the two actions are equal. \square

3.2 Coquasitriangular Hopf Algebras

Let $\pi : G \rightarrow H$ be a quantum principal bundle with $\Omega^1(M)$ an object in ${}^G_M\text{Mod}_0$. If H has coquasitriangular structure r , then by §2.1.3 the comodule $V_M \in \text{Mod}^H$ has an associated braiding $\sigma : V_M \otimes V_M \rightarrow V_M \otimes V_M$, and hence an associated Nichols algebra. In practice this braided can prove to be easier to calculate than the Yetter–Drinfeld braiding.

As noted earlier, a coaction of V_M extends via the adjoint coaction to a coaction $\text{Ad}_R : \Lambda_G^1 \rightarrow \Lambda_G^1$. With respect to r this also gives a braiding for Λ^1 extending σ . In practice this extension can prove easier again to work with. The following lemma gives an explicit expression for this operator. Note that in order to lighten notation we denote

$$r_\pi : G \otimes G \rightarrow \mathbb{C}, \quad f \otimes g \mapsto r(\pi(f) \otimes \pi(g)),$$

and the corresponding definition for $\bar{r}_\pi i$.

Lemma 3.2 *For a quantum principal bundle with total cotangent space Λ^1 , the braiding induced by the coquasi-triangular structure r acts as*

$$\sigma([f] \otimes [g]) = [g_{(3)}] \otimes [f_{(3)}] \bar{r}_\pi(f_{(2)} \otimes g_{(4)}) r_\pi(f_{(1)} \otimes g_{(2)}) r_\pi(f_{(4)} \otimes g_{(5)}) r_\pi(f_{(5)} \otimes S(g_{(1)})).$$

Proof. This follows directly from the calculation

$$\begin{aligned} \sigma([f] \otimes [g]) &= [g_{(2)}] \otimes [f_{(2)}] (r_\pi(S(f_{(1)})f_{(3)} \otimes S(g_{(1)})g_{(3)})) \\ &= [g_{(3)}] \otimes [f_{(2)}] r_\pi(S(f_{(1)}) \otimes S(g_{(2)})g_{(4)}) r_\pi(f_{(3)} \otimes S(g_{(1)})g_{(5)}) \\ &= [g_{(3)}] \otimes [f_{(3)}] r_\pi(S(f_{(2)}) \otimes g_{(4)}) r_\pi(S(f_{(1)}) \otimes S(g_{(2)})) r_\pi(f_{(4)} \otimes g_{(5)}) r_\pi(f_{(5)} \otimes S(g_{(1)})) \\ &= [g_{(3)}] \otimes [f_{(3)}] \bar{r}_\pi(f_{(2)} \otimes g_{(4)}) r_\pi(f_{(1)} \otimes g_{(2)}) r_\pi(f_{(4)} \otimes g_{(5)}) r_\pi(f_{(5)} \otimes S(g_{(1)})). \end{aligned}$$

\square

4 The Quantum Grassmannians

In this section we recall the definition of the quantum Grassmannians and quantum principal bundle induced by the quantum Killing form. It is observed that the bundle satisfies the requirements of Theorem 3.1, and so, has an associated Yetter–Drinfeld. The corresponding Nichols algebras of the first order anti-holomorphic calculus is then examined and shown to be equal to its maximal prolongation. Moreover, it is shown to be non-diagonal and of Hecke type.

4.1 The Quantum Special Unitary Group $\mathbb{C}_q[SU_{n+1}]$

We begin by fixing notation and recalling the various definitions and constructions needed to introduce the quantum unitary group and the quantum special unitary group. (Where proofs or basic details are omitted we refer the reader to [4, §9.2].)

For $q \in (0, 1]$ and $\nu := q - q^{-1}$, let $\mathbb{C}_q[GL_N]$ be the quotient of the free noncommutative algebra $\mathbb{C}\langle u_j^i, \det^{-1} \mid i, j = 1, \dots, N \rangle$ by the ideal generated by the elements

$$\begin{aligned} u_k^i u_k^j - q u_k^j u_k^i, & \quad u_i^k u_j^k - q u_j^k u_i^k, & \quad 1 \leq i < j \leq N, 1 \leq k \leq N; \\ u_l^i u_k^j - u_k^j u_l^i, & \quad u_k^i u_l^j - u_l^j u_k^i - \nu u_l^i u_k^j, & \quad 1 \leq i < j \leq N, 1 \leq k < l \leq N; \\ \det_N \det_N^{-1} - 1, & \quad \det_N^{-1} \det_N - 1, \end{aligned}$$

where \det_N , the *quantum determinant*, is the element

$$\det_N := \sum_{\pi \in S_N} (-q)^{\ell(\pi)} u_{\pi(1)}^1 u_{\pi(2)}^2 \cdots u_{\pi(N)}^N$$

with summation taken over all permutations π of the set $\{1, \dots, N\}$, and $\ell(\pi)$ is the number of inversions in π . As is well-known, \det_N is a central and grouplike element of the bialgebra.

A bialgebra structure on $\mathbb{C}_q[GL_N]$ with coproduct Δ , and counit ε , is uniquely determined by $\Delta(u_j^i) := \sum_{k=1}^N u_k^i \otimes u_j^k$; $\Delta(\det_N^{-1}) = \det_N^{-1} \otimes \det_N^{-1}$; and $\varepsilon(u_j^i) := \delta_{ij}$; $\varepsilon(\det_N^{-1}) = 1$. Moreover, we can endow $\mathbb{C}_q[GL_N]$ with a Hopf algebra structure by defining

$$S(\det_N^{-1}) = \det_N, \quad S(u_j^i) = (-q)^{i-j} \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u_{\pi(l_1)}^{k_1} u_{\pi(l_2)}^{k_2} \cdots u_{\pi(l_{N-1})}^{k_{N-1}} \det_N^{-1},$$

where $\{k_1, \dots, k_{N-1}\} = \{1, \dots, N\} \setminus \{j\}$, and $\{l_1, \dots, l_{N-1}\} = \{1, \dots, N\} \setminus \{i\}$ as ordered sets. A Hopf $*$ -algebra structure is determined by $(\det_N^{-1})^* = \det_N$, and $(u_j^i)^* = S(u_i^j)$. We denote the Hopf $*$ -algebra by $\mathbb{C}_q[U_N]$, and call it the *quantum unitary group of order N* . We denote the Hopf $*$ -algebra $\mathbb{C}_q[U_N]/\langle \det_N - 1 \rangle$ by $\mathbb{C}_q[SU_N]$, and call it the *quantum special unitary group of order N* .

4.2 The Corepresentations of $\mathbb{C}_q[SU_n]$

A *Young diagram* is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. Young diagrams with m rows are clearly equivalent to *dominant weights* of order m , which is to say elements

$$\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{N}^m, \quad \text{such that } \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m.$$

We denote the set of dominant weights of order m by $\text{Dom}(m)$. A *semi-standard tableau* of shape λ with labels in $\{1, \dots, n\}$ is a collection $\mathbf{T} = \{T_{rs}\}_{r,s}$ of elements of $\{1, \dots, n\}$ indexed by the boxes of the corresponding Young diagram and satisfying, whenever defined, the inequalities

$$T_{r-1,s} < T_{r,s}, \quad T_{r,s-1} \leq T_{r,s}.$$

We denote by $\text{SSTab}_n(\lambda)$ the set of all semi-standard tableau for a given dominant weight λ . The *standard monomial* associated to a semi-standard tableau \mathbf{T} is

$$z^{\mathbf{T}} := z^{T_1} \dots z^{T_{\lambda_1}} \in \mathbb{C}_q[SU_n],$$

where $T_s := \{T_{1,s}, \dots, T_{m_s,s}\}$ as an ordered set, for $1 \leq s \leq \lambda_1$, and m_s is the length of the s^{th} column. It can be shown [16, §2] that the elements $z^{\mathbf{T}}$ are linearly independent, and that the space

$$V_L(\lambda) := \text{span}_{\mathbb{C}}\{z^{\mathbf{T}} \mid \mathbf{T} \in \text{SSTab}_n(\lambda)\}$$

is an irreducible left $\mathbb{C}_q[SU_n]$ -comodule. Moreover, every irreducible left comodule of $\mathbb{C}_q[SU_n]$ is isomorphic to $V(\lambda)$, for some λ . Similarly, for $z_{\mathbf{T}} := z_{T_1} \dots z_{T_l} \in \mathbb{C}_q[SU_n]$, the space $V_R(\lambda) := \text{span}_{\mathbb{C}}\{z_{\mathbf{T}} \mid \mathbf{T} \in \text{SSTab}_n(\lambda)\}$ is an irreducible right $\mathbb{C}_q[SU_n]$ -comodule and all irreducible right $\mathbb{C}_q[SU_n]$ -comodules are of this form. \square

Let $\mathbb{C}[\mathbb{T}^n]$ be the commutative polynomial algebra generated by t_k, t_k^{-1} , for $k = 1, \dots, n$, satisfying the obvious relation $t_k t_k^{-1} = 1$. We can give $\mathbb{C}[\mathbb{T}^n]$ the structure of a Hopf algebra by defining a coproduct, counit and antipode according to $\Delta(t_i) = t_i \otimes t_i$, $\varepsilon(t_k) = 1$ and $S(t_k) = t_k^{-1}$. (Note that $\mathbb{C}[\mathbb{T}^1] \simeq \mathbb{C}[U_1]$.)

A basis of $\mathbb{C}[\mathbb{T}^n]$ is given by

$$\{t^\lambda := t_1^{l_1} \dots t_n^{l_n} \mid \lambda = (l_1, \dots, l_n) \in \mathbb{Z}^n\}.$$

Since each basis element is grouplike, a $\mathbb{C}[\mathbb{T}^n]$ -comodule structure is equivalent to a \mathbb{Z}^n -grading. We call the homogeneous elements of such a grading *weight vectors*.

Let $\tau : \mathbb{C}_q[U_n] \rightarrow \mathbb{C}[\mathbb{T}^{n-1}]$ be the surjective Hopf $*$ -algebra map defined by setting

$$\tau(u_j^i) = \delta_{ij} t_i, \quad \tau(u_n^n) = t_1^{-1} \dots t_{n-1}^{-1}, \quad \tau(\det_n^{-1}) = 1, \quad i, j = 1, \dots, n; (i, j) \neq (n, n).$$

For any left $\mathbb{C}_q[U_n]$ -comodule V , a left $\mathbb{C}[\mathbb{T}^{n-1}]$ -comodule structure on V is defined by $\Delta_{L,\tau} := (\tau \otimes \text{id})\Delta_L$.

4.3 The Quantum Grassmannians $\mathbb{C}_q[\mathbb{C}P^n]$

Let $\alpha_n : \mathbb{C}_q[SU_n] \rightarrow \mathbb{C}_q[U_{n-1}]$ be the Hopf $*$ -algebra map defined by $\alpha_n(u_1^1) := \det_{n-1}^{-1}$; $\alpha_n(u_i^1) = \alpha_n(u_1^i) := 0$, for $i \neq 1$; and $\alpha_n(u_j^i) := u_{j-1}^{i-1}$, for $i, j = 2, \dots, n$. Moreover, let $\alpha'_n : \mathbb{C}_q[SU_n] \rightarrow \mathbb{C}_q[U_{n-1}]$ be the Hopf algebra map defined by $\alpha'_n(u_1^1) := \det_{n-1}^{-1}$; $\alpha'_n(u_i^n) = \alpha'_n(u_n^i) := 0$, for $i \neq 1$; and $\alpha'_n(u_j^i) := u_j^i$, for $i, j = 1, \dots, n-1$.

Definition 4.1. The *quantum (n, r) -Grassmannian* $\mathbb{C}_q[\text{Gr}_{n,r}]$ is the quantum homogeneous space associated to the surjective Hopf $*$ -algebra map $\pi_{n,r} : \mathbb{C}_q[SU_n] \rightarrow \mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[U_{n-r}]$ for

$$\pi_{n,r} := (\alpha_{r-1} \circ \dots \circ \alpha_n) \otimes (\beta_{n-r-1} \circ \dots \circ \beta_n) \circ \Delta.$$

We now observe that using the theory of bicovariant splitting maps, we can formulate an alternative proof of a result of Müller and Schneider [14, Corollary 1.5.1].

Lemma 4.2 *Every quantum homogeneous space $\pi : G \rightarrow H$ with H cosemisimple is a principal comodule algebra.*

Proof. Let $\pi : G \rightarrow H$ be a Hopf algebra surjection with quantum homogeneous space $M := G^{\text{co}H}$. By assumption H is cosemisimple, then since $\pi : G \rightarrow H$ is a bicomodule map, it admits a bicomodule splitting. Hence, by Proposition 2.3 it is a principal comodule algebra. \square

Since $\mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[U_{n-r}]$ is the product of two cosemisimple Hopf algebras, and hence itself cosemisimple, the discussion in §2.1.3 implies that $\mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n]$ is a principal comodule algebra.

4.4 The Quantum Principal Bundle

Recall that a first-order calculus is called *irreducible* if it contains no non-trivial sub-bimodule.

Theorem 4.3 [7, §2] *There exist exactly two non-isomorphic irreducible left-covariant first-order differential calculi of finite dimension over $\mathbb{C}_q[\text{Gr}_{n,k}]$. Moreover, each is an object in the subcategory ${}^G_M\text{Mod}_0$.*

We call the direct sum of these two calculi the *Heckenberger–Kolb* calculus of $\mathbb{C}_q[\text{Gr}_{n,k}]$, and denote it by $\Omega_q^1(\text{Gr}_{n,r})$. Classically, these two calculi are the holomorphic and anti-holomorphic parts of the complexified cotangent bundle of $\mathbb{C}_q[\text{Gr}_{n,r}]$.

Consider now the ideal $I_r \subseteq \mathbb{C}_q[SU_n]$ defined by

$$I_r := \ker(Q) + \{u_j^i \mid j = r+1, \dots, n\},$$

where Q is the quantum Killing form associated to the coquasitriangular structure r . We denote the corresponding left covariant calculus by $\Omega_q^1(SU_n; r)$.

Theorem 4.4 [13] *The calculus $\Omega_q^1(SU_n; r)$ restricts to $\Omega_q^1(Gr_{n,r})$ on $\mathbb{C}_q[Gr_{n,r}]$. Moreover, it induces a quantum principal bundle structure on the Hopf-Galois extension $\mathbb{C}_q[Gr_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n]$.*

As a direct consequence of this result and Theorem 3.1 we get the following result.

Corollary 4.5 *Any bicovariant splitting map $i : \mathbb{C}_q[Gr_{n,r}] \rightarrow \mathbb{C}_q[SU_n]$ induces the structure of a $\mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[U_{n-r}]$ -Yetter-Drinfeld module on the cotangent space of $\Omega_q^1(\mathbb{C}_q[Gr_{n,r}])$.*

4.5 The Braidings

4.5.1 Coaquasitriangular Braiding

For the braiding induced by the coaquasitriangular r , we find it useful to scale it by a factor of q^{-1} and denote the new braiding by σ .

Lemma 4.6 *The braiding acts as*

$$\begin{aligned} \sigma(e_i^- \otimes e_i^-) &= e_i^- \otimes e_i^-, & \sigma(e_i^- \otimes e_j^-) &= q^{1-\delta_{ij}} e_j^- \otimes e_i^-, \\ \sigma(e_j^- \otimes e_i^-) &= q^{1-\delta_{ij}} e_i^- \otimes e_j^- + q^{-\delta_{ij}} (1 - q^{-2}) e_j^- \otimes e_i^-. & i = 1, \dots, n, i \neq j. \end{aligned}$$

Proof. Applying the formula from Lemma 3.2, we get that

$$\begin{aligned} \sigma(u_i^1 \otimes u_k^1) &= \sum [u_y^x] \otimes [u_c^b] \bar{r}_\pi(u_b^a \otimes u_z^y) r_\pi(u_a^1 \otimes u_x^w) r_\pi(u_d^c \otimes u_k^z) r_\pi(u_i^d \otimes S(u_w^1)) \\ &= \sum [u_y^1] \otimes [u_c^1] \bar{r}_\pi(u_1^1 \otimes u_z^y) r_\pi(u_1^1 \otimes u_1^1) r_\pi(u_d^c \otimes u_k^z) r_\pi(u_i^d \otimes S(u_1^1)) \\ &= \sum [u_z^1] \otimes [u_c^1] r_\pi(u_i^c \otimes u_k^z) \\ &= \sum [u_z^1] \otimes [u_c^1] R_{ik}^{cz}. \end{aligned}$$

For the case $i \geq k$, this gives

$$\sigma(u_i^1 \otimes u_k^1) = [u_k^1] \otimes [u_i^1] R_{ik}^{ik} = q^{-\delta_{ik}} [u_k^1] \otimes [u_i^1].$$

For $i < k$, we have

$$\begin{aligned} \sigma(u_i^1 \otimes u_k^1) &= [u_k^1] \otimes [u_i^1] R_{ik}^{ik} + [u_i^1] \otimes [u_k^1] R_{ik}^{ki} \\ &= q^{-\delta_{ik}} [u_k^1] \otimes [u_i^1] + (q - q^{-1}) [u_i^1] \otimes [u_k^1]. \quad \square \end{aligned}$$

We now use A_2 , along with the corepresentation theory of $\mathbb{C}_q[SU_n]$ presented in §4.2, to decompose $(V^{(0,1)})^{\otimes 2}$ into irreducible subcomodules.

Lemma 4.7 *It holds that*

$$\begin{aligned}\ker(A_2) &= \text{span}_{\mathbb{C}}\{e_i^- \otimes e_i^-, e_i^- \otimes e_j^- - qe_j^- \otimes e_i^- \mid i, j = 1, \dots, n; i < j\} \\ \text{Im}(A_2) &= \text{span}_{\mathbb{C}}\{e_i^- \otimes e_j^- + q^{-1}e_j^- \otimes e_i^- \mid i, j = 1, \dots, n; i < j\}.\end{aligned}$$

Moreover, both $\ker(A_2)$ and $\text{Im}(A_2)$ are irreducible $\mathbb{C}_q[U_{n-1}]$ -comodules, and the decomposition $(V^{(0,1)})^{\otimes 2} \simeq \ker(A_2) \oplus \text{Im}(A_2)$ is the unique decomposition of $(V^{(0,1)})^{\otimes 2}$ into irreducible subcomodules.

Proof. The descriptions of $\ker(A_2)$ and $\text{Im}(A_2)$ are elementary consequences of the formulae for the braiding given in the above lemma.

We now use a weight argument to show irreducibility of the two comodules. First note that

$$\begin{aligned}\Delta_{\tau}(e_i^- \otimes e_j^-) &= \Delta_{\tau}([u_i^1] \otimes [u_j^1]) \\ &= \sum_{a,b,c,d=1}^n [u_b^a] \otimes [u_d^c] \otimes \tau(S(u_a^1)u_j^b)S(u_c^1)u_j^d \\ &= [u_i^1] \otimes [u_j^1] \otimes \tau(S(u_1^1)u_i^i)S(u_1^1)u_j^j \\ &= [u_i^1] \otimes [u_j^1] \otimes \tau(u_i^i)\tau(u_j^j).\end{aligned}$$

Hence, we see that the highest weight vector of $\ker(A_2)$ is $e_n^- \otimes e_n^-$ with highest weight $2n$. Since the comodule with highest weight $2n$ has dimension $n + \binom{n}{2}$, it must hold that $\ker(A_2)$ is irreducible. A similar argument shows that $\text{Im}(A_2)$ is irreducible.

It is clear that $(V^{(0,1)})^{\otimes 2} \simeq \ker(A_2) \oplus \text{Im}(A_2)$. Moreover, since both comodules are non-isomorphic, having different dimensions, this is the only decomposition into irreducible subcomodules. \square

This implies the result whose result is elementary and omitted.

Corollary 4.8 *There does not exist a non-trivial diagonal braiding $c : V^{(0,1)} \otimes V^{(0,1)} \rightarrow V^{(0,1)} \otimes V^{(0,1)}$. Moreover, every braiding is proportional to a braiding of Hecke type.*

4.6 The Anti-Holomorphic Nichols Algebra

Theorem 4.9 *The braidings σ_i and σ_r are not proportional. However, there exists a scaled braidings σ'_i such that*

$$\mathfrak{B}_{\sigma'_i}(V) = \mathfrak{B}_{\sigma'_r}(V) = V^{(0,\bullet)}.$$

Proof. We begin by showing that $\mathfrak{B}_{\sigma'_i}(V) = V^{(0,\bullet)}$. Since $I^{(2)}$ is a proper subcomodule of $V^{\otimes 2}$, it follows from Lemma 4.7 that it is equal to $\ker(A_1)$ or $\text{Im}(A_2)$. Since we know that $\dim(V^2) = \binom{n}{2}$ it must be that $I^{(2)} = \ker(A_2)$. It now follows from Proposition 2.1 and Lemma 4.8 that $\mathfrak{B}_{\sigma'_i}(V) = V^{(0,\bullet)}$.

Let us now calculate the action of σ_i on the element $e_j^- \otimes e_i^-$. By definition

$$\begin{aligned}\sigma_i([u_j^1] \otimes [u_i^1]) &= \sum_{a,b=1}^n [u_b^a] \otimes ([u_j^1] \triangleleft (\pi(S(u_a^1))\pi(u_i^b))) \\ &= \sum_{b=1}^n [u_b^1] \otimes ([u_j^1] \triangleleft (u_{i-1}^{b-1} \det_{n-1})) \\ &= \sum_{b=1}^n [u_b^1] \otimes ([u_j^1 u_i^b] \triangleleft \det_{n-1}).\end{aligned}$$

It follows from [18, Proposition 3.3] that $[u_j^1 u_i^b] \neq 0$ if and only if $b = i$, whereupon $[u_j^1 u_i^i] = q^{\delta_{i1}} [u_j^1]$. Hence, we see that

$$\begin{aligned}\sigma_i([u_j^1] \otimes [u_i^1]) &= [u_i^1] \otimes ([u_j^1 u_i^i] \triangleleft \det_{n-1}) \\ &= q^{\delta_{i1}} [u_i^1] \otimes ([u_j^1] \triangleleft \det_{n-1}) \\ &= q^{\delta_{i1}} [u_i^1] \otimes [u_j^1 S(u_1^1)] \\ &= q^{\delta_{i1}} [u_i^1] \otimes [u_j^1].\end{aligned}$$

Hence, σ_i is not proportional to σ_r . However, σ_i can clearly be scaled so as to coincide with σ_r on $\ker(A_2)$. Thus the kernel of A_2 for this scaled braiding will contain $I^{(2)}$. Using the same argument as for σ_r , one can now show that $\mathfrak{B}_{\sigma'_i}(V) = V^{(0,\bullet)}$. \square

Corollary 4.10 *The exterior algebra $V^{(0,\bullet)}$ is a Koszul algebra and a braided Hopf algebra in the category of Yetter–Drinfeld modules over $\mathbb{C}_q[U_{n-1}]$.*

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